

INTEGRATION BY PARTS (TABLE METHOD)

Suppose you want to evaluate $\int x^2 \cos 3x \, dx$ using integration by parts.

Using the $\int u \, dv$ notation, we get

$$u = x^2 \quad dv = \cos 3x \, dx$$

$$du = 2x \, dx \quad v = \frac{1}{3} \sin 3x$$

$$\text{So,} \quad \int x^2 \cos 3x \, dx = x^2 \left(\frac{1}{3} \sin 3x \right) - \int 2x \left(\frac{1}{3} \sin 3x \right) dx \quad \text{or} \quad x^2 \left(\frac{1}{3} \sin 3x \right) - \int \frac{2}{3} x \sin 3x \, dx$$

We see that it is necessary to perform integration by parts a 2nd time.

If we do that without multiplying together and simplifying or factoring out any constants, we get

$$u = 2x \quad dv = \frac{1}{3} \sin 3x \, dx$$

$$du = 2 \, dx \quad v = -\frac{1}{9} \cos 3x$$

$$\text{So,} \quad \int x^2 \cos 3x \, dx = x^2 \left(\frac{1}{3} \sin 3x \right) - \left[2x \left(-\frac{1}{9} \cos 3x \right) - \int 2 \left(-\frac{1}{9} \cos 3x \right) dx \right]$$

At this point, the remaining integral is simple enough to finish without integration by parts (★).

But suppose we perform integration by parts a 3rd time, again without changing the constants, then we get

$$u = 2 \quad dv = -\frac{1}{9} \cos 3x \, dx$$

$$du = 0 \, dx \quad v = -\frac{1}{27} \sin 3x$$

$$\text{So,} \quad \int x^2 \cos 3x \, dx = x^2 \left(\frac{1}{3} \sin 3x \right) - \left\{ 2x \left(-\frac{1}{9} \cos 3x \right) - \left[2 \left(-\frac{1}{27} \sin 3x \right) - \int 0 \left(-\frac{1}{27} \sin 3x \right) dx \right] \right\}$$

Since the integrand in the final integral is now 0, its anti-derivative is simply a constant (C).

We can let $C = 0$ for now, and add a constant of integration later.

$$\text{So,} \quad \int x^2 \cos 3x \, dx = x^2 \left(\frac{1}{3} \sin 3x \right) - \left[2x \left(-\frac{1}{9} \cos 3x \right) - 2 \left(-\frac{1}{27} \sin 3x \right) \right] + C$$

$$\int x^2 \cos 3x \, dx = x^2 \left(\frac{1}{3} \sin 3x \right) - 2x \left(-\frac{1}{9} \cos 3x \right) + 2 \left(-\frac{1}{27} \sin 3x \right) + C$$

$$\int x^2 \cos 3x \, dx = \frac{1}{3} x^2 \sin 3x + \frac{2}{9} x \cos 3x - \frac{2}{27} \sin 3x + C$$

$$\int x^2 \cos 3x \, dx = \frac{1}{27} ((9x^2 - 2) \sin 3x + 6x \cos 3x) + C$$

(★) If we had skipped the final “unnecessary” integration by parts, we would have done these last 3 lines anyway.

Now, in the traditional method,

you would probably want to simplify and factor all the constants before each new integration by parts, and skip the last integration by parts.

However, by not doing so,

we can actually perform the entire multi-step integration by parts inside a single table.

To understand how the table method of integration by parts works,
 first notice that we didn't really change $du = 2x \, dx$ and $v = \frac{1}{3} \sin 3x$
 from the result of the 1st integration by parts,
 and just reused them as $u = 2x$ and $dv = \frac{1}{3} \sin 3x \, dx$
 for the 2nd integration by parts.

That means that we could “eliminate” writing that repeated line.

This also applies to $du = 2 \, dx$ and $v = -\frac{1}{9} \cos 3x$

from the 2nd integration by parts

being reused as $u = 2$ and $dv = -\frac{1}{9} \cos 3x \, dx$

for the 3rd integration by parts. And so on.

So the entire integration by parts could have been compressed into the following table.

<u>u</u>	<u>dv</u>	
x^2	$\cos 3x$	← product = integrand from 1 st integration by parts
DIFFERENTIATE ↓	↓ ANTIDIFFERENTIATE	
$2x$	$\frac{1}{3} \sin 3x$	← product = integrand from 2 nd integration by parts
DIFFERENTIATE ↓	↓ ANTIDIFFERENTIATE	
2	$-\frac{1}{9} \cos 3x$	← product = integrand from 3 rd integration by parts
DIFFERENTIATE ↓	↓ ANTIDIFFERENTIATE	
0	$-\frac{1}{27} \sin 3x$	← product = integrand from 4 th integration by parts

The **downward diagonal products** (one row in the u column $\times$ the next row in the dv column)
 are the results of the uv portion of the integration by parts formula.

However, because of the negative in the $-\int v \, du$ part of the formula,
 which distributes through each successive integration by parts,
 the products must be **alternately added and subtracted**.

In other words, to read the antiderivative from the table above:

$$\int x^2 \cos 3x \, dx = x^2 \left(\frac{1}{3} \sin 3x \right) - 2x \left(-\frac{1}{9} \cos 3x \right) + 2 \left(-\frac{1}{27} \sin 3x \right) + C = \frac{1}{27} ((9x^2 - 2) \sin 3x + 6x \cos 3x) + C$$

The same result with much less writing.

Notice that the fourth row of the u column (0) is ignored.
 We could have viewed it as a factor of the integrand in the 4th integration by parts,
 which means its antiderivative would have been 0 (or C).
 Or we could have viewed it as a factor of the next downward diagonal product,
 which also means its corresponding antiderivative would have been 0 .
 In either case, it contributes nothing to the antiderivative.

However, because of that 0 in the u column,
 all following derivatives in the u column would also have been 0 .
 So we can stop taking any more derivatives or antiderivatives.
 In fact, the 0 in the u column tells us to stop filling out the table,
 and start collecting the answer.

The organizational structure makes it a much faster and less error-prone method which requires a lot less writing. It also makes it easier to find mistakes later on.

This method can be used for antiderivatives of the types

$$\int (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) e^{kx} dx$$

where n is a positive integer, and k is a constant

$$\int (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) \sin kx dx$$

where n is a positive integer, and k is a constant

$$\int (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) \cos kx dx$$

where n is a positive integer, and k is a constant

$$\int (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) \sinh kx dx$$

where n is a positive integer, and k is a constant

$$\int (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) \cosh kx dx$$

where n is a positive integer, and k is a constant

$$\int (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) (mx + b)^k dx$$

where n is a positive integer, and m and b are constants,

and k is a constant such that $k \neq -1$ (and never becomes -1 during the repeated antidifferentiations)

If you put the polynomial factor into the u column, then by repeatedly differentiating it, you eventually get 0 in the u column.

In the meantime, the dv column can be antidifferentiated repeatedly using basic techniques.



The method cannot be used directly for antiderivatives of the types

$$\int e^{ax} \sin kx \, dx \quad \text{where } a \text{ and } k \text{ are constants}$$

$$\int e^{ax} \cos kx \, dx \quad \text{where } a \text{ and } k \text{ are constants}$$

but it can be used for part of the integration.

Consider $\int e^{2x} \cos 3x \, dx$.

u		dv	
$\cos 3x$		e^{2x}	product = integrand from 1 st integration by parts
DIFFERENTIATE ↓		ANTIDIFFERENTIATE ↓	
$-3 \sin 3x$	+	$\frac{1}{2} e^{2x}$	product = integrand from 2 nd integration by parts
DIFFERENTIATE ↓		ANTIDIFFERENTIATE ↓	
$-9 \cos 3x$	-	$\frac{1}{4} e^{2x}$	product = integrand from 3 rd integration by parts

After differentiating/antidifferentiating twice,

it becomes clear we will never get 0 in the u column.

However, we have arrived at a row

that is essentially the same as the original row (except for constant factors).

That is enough, because remembering that the product across the 3rd row is an integrand, we get

$$\int e^{2x} \cos 3x \, dx = (\cos 3x) \left(\frac{1}{2} e^{2x} \right) - (-3 \sin 3x) \left(\frac{1}{4} e^{2x} \right) + \int (-9 \cos 3x) \left(\frac{1}{4} e^{2x} \right) dx$$

$$\int e^{2x} \cos 3x \, dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cos 3x \, dx$$

The original integral appears on both sides of the equation, so we can isolate it.

$$\int e^{2x} \cos 3x \, dx + \frac{9}{4} \int e^{2x} \cos 3x \, dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x$$

$$\frac{13}{4} \int e^{2x} \cos 3x \, dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x$$

$$\int e^{2x} \cos 3x \, dx = \frac{4}{13} \left(\frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x \right)$$

$$\int e^{2x} \cos 3x \, dx = \frac{2}{13} e^{2x} \cos 3x + \frac{3}{13} e^{2x} \sin 3x + C$$

$$\int e^{2x} \cos 3x \, dx = \frac{1}{13} e^{2x} (2 \cos 3x + 3 \sin 3x) + C$$

This technique works whenever repeated integrations by parts

results in integrands which are constant multiples of the original integral, which includes antiderivatives of the types

(where k and m are non-zero constants and (★) where $k \neq m$)

$$\int \sin kx \sin mx \, dx \text{ (★)}$$

$$\int \sin kx \cos mx \, dx$$

$$\int \cos kx \cos mx \, dx \text{ (★)}$$

$$\int \sin kx \sinh mx \, dx$$

$$\int \sin kx \cosh mx \, dx$$

$$\int \cos kx \sinh mx \, dx$$

$$\int \cos kx \cosh mx \, dx$$

The technique also works for antiderivatives of the types
(where k and m are non-zero constants and (★) where $k \neq m$)

$$\begin{array}{lll} \int e^{ax} \sinh kx \, dx & \int e^{ax} \cosh kx \, dx & \\ \int \sinh kx \sinh mx \, dx \text{ (★)} & \int \sinh kx \cosh mx \, dx & \int \cosh kx \cosh mx \, dx \text{ (★)} \end{array}$$

However, these could all be done much more easily
by replacing the hyperbolic functions with their exponential definitions,
simplifying the integrand,
and then antidifferentiating without integration by parts.

By the way, the original integral could also have been done by reversing the initial choice of u and dv .

u		dv	
e^{2x}		$\cos 3x$	← product = integrand from 1 st integration by parts
DIFFERENTIATE ↓	+	↓ ANTIDIFFERENTIATE	
$2e^{2x}$		$\frac{1}{3} \sin 3x$	← product = integrand from 2 nd integration by parts
DIFFERENTIATE ↓	-	↓ ANTIDIFFERENTIATE	
$4e^{2x}$	+	$-\frac{1}{9} \cos 3x$	← product = integrand from 3 rd integration by parts

$$\int e^{2x} \cos 3x \, dx = e^{2x} \left(\frac{1}{3} \sin 3x \right) - 2e^{2x} \left(-\frac{1}{9} \cos 3x \right) + \int 4e^{2x} \left(-\frac{1}{9} \cos 3x \right) dx$$

$$\int e^{2x} \cos 3x \, dx = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x \, dx$$

$$\int e^{2x} \cos 3x \, dx + \frac{4}{9} \int e^{2x} \cos 3x \, dx = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x$$

$$\frac{13}{9} \int e^{2x} \cos 3x \, dx = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x$$

$$\int e^{2x} \cos 3x \, dx = \frac{9}{13} \left(\frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x \right)$$

$$\int e^{2x} \cos 3x \, dx = \frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x + C$$

$$\int e^{2x} \cos 3x \, dx = \frac{1}{13} e^{2x} (3 \sin 3x + 2 \cos 3x) + C$$

The result and amount of work is the same as the original choice.

However, mistakes are more common when antidifferentiating sines and cosines than when differentiating them.

[Mistakes are much less common when antidifferentiating or differentiating exponentials.]

You could also have differentiated u and antidifferentiated dv 4, 6 or any even number of times
and gotten back to a constant multiple of the original integrand.

If you had done so, and completed the algebra in a similar way as shown above,
you would gotten the same answer,
but with a lot more (unnecessary) work.

And remember that doing unnecessary work increases the chances of making mistakes.

The method can be used for antiderivatives of the type

$$\int x^p (\ln x)^n dx \quad \text{where } p \text{ is a constant and } n \text{ is a positive integer}$$

with a not-so-slight modification to the process.

Consider $\int x^{\frac{1}{3}} (\ln x)^2 dx$

First, let's consider what doesn't work well.

<u>u</u>	<u>dv</u>
$x^{\frac{1}{3}}$	$(\ln x)^2$
DIFFERENTIATE ↓	ANTIDIFFERENTIATE
$\frac{1}{3}x^{-\frac{2}{3}}$	?

← product = integrand from 1st integration by parts

We cannot put $\ln x$ into the dv column because we don't know its antiderivative right now.

Even if you know the antiderivative of $\ln x$, it is more complicated than $\ln x$ itself, which goes against the basic rule of integration by parts, that each successive integrand should not be more complicated than the previous.

So we are forced to put $\ln x$ into the u column.

<u>u</u>	<u>dv</u>
$(\ln x)^2$	$x^{\frac{1}{3}}$
DIFFERENTIATE ↓	ANTIDIFFERENTIATE
$\frac{2(\ln x)}{x}$	$\frac{3}{4}x^{\frac{4}{3}}$

← product = integrand from 1st integration by parts

← product = integrand from 2nd integration by parts

This does not look promising, because we know the next row in the u column will involve the quotient rule, which again goes against the basic rule of integration of parts, that each successive integrand should not be more complicated than the previous.

So, perhaps like the previous example, we need to rewrite the original integrand using what we've done so far.

$$\int x^{\frac{1}{3}} (\ln x)^2 dx = (\ln x)^2 \left(\frac{3}{4} x^{\frac{4}{3}} \right) - \int \frac{2(\ln x)}{x} \left(\frac{3}{4} x^{\frac{4}{3}} \right) dx$$

$$\int x^{\frac{1}{3}} (\ln x)^2 dx = \frac{3}{4} x^{\frac{4}{3}} (\ln x)^2 - \int \frac{3}{2} x^{\frac{1}{3}} \ln x dx$$

The new integral is simpler than the original because the power of the $\ln x$ has been reduced by 1.

In fact, you might guess that if we repeat the integration by parts a second time, we can reduce the power of the $\ln x$ by 1 more, to 0.

We can do that, but if we're going to do a 2nd integration by parts, shouldn't there be a way to do it without going into and out of the table method repeatedly?

There is, and it amounts to rebalancing the u and dv columns in the middle of the repeated differentiations and antidifferentiations.

$\frac{u}{}$	$\frac{dv}{}$	
$(\ln x)^2$	$\frac{1}{x^3}$	↔ product = integrand from 1 st integration by parts
DIFFERENTIATE ↓	↓	ANTIDIFFERENTIATE
$\frac{2(\ln x)}{x}$	$\frac{3}{4}x^{\frac{4}{3}}$	↔ product = integrand from 2 nd integration by parts
MULTIPLY BY x		MULTIPLY BY x^{-1}
$2(\ln x)$	$\frac{3}{4}x^{\frac{1}{3}}$	↔ product = integrand from 2 nd integration by parts (same as line above)

The rebalancing of the table allows us to continue differentiating the u column without introducing a complicated expression into the table.

$\frac{u}{}$	$\frac{dv}{}$	
$(\ln x)^2$	$\frac{1}{x^3}$	↔ product = integrand from 1 st integration by parts
DIFFERENTIATE ↓	↓	ANTIDIFFERENTIATE
$\frac{2(\ln x)}{x}$	$\frac{3}{4}x^{\frac{4}{3}}$	↔ product = integrand from 2 nd integration by parts
MULTIPLY BY x		MULTIPLY BY x^{-1}
$2(\ln x)$	$\frac{3}{4}x^{\frac{1}{3}}$	↔ product = integrand from 2 nd integration by parts (same as line above)
DIFFERENTIATE ↓	↓	ANTIDIFFERENTIATE
$\frac{2}{x}$	$\frac{9}{16}x^{\frac{4}{3}}$	↔ product = integrand from 3 rd integration by parts

This last product can be integrated without integration by parts, but we can complete the whole process in the table by rebalancing once more.

$\frac{u}{}$	$\frac{dv}{}$	
$(\ln x)^2$	$\frac{1}{x^3}$	↔ product = integrand from 1 st integration by parts
DIFFERENTIATE ↓	↓	ANTIDIFFERENTIATE
$\frac{2(\ln x)}{x}$	$\frac{3}{4}x^{\frac{4}{3}}$	↔ product = integrand from 2 nd integration by parts
MULTIPLY BY x		MULTIPLY BY x^{-1}
$2(\ln x)$	$\frac{3}{4}x^{\frac{1}{3}}$	↔ product = integrand from 2 nd integration by parts (same as line above)
DIFFERENTIATE ↓	↓	ANTIDIFFERENTIATE
$\frac{2}{x}$	$\frac{9}{16}x^{\frac{4}{3}}$	↔ product = integrand from 3 rd integration by parts
MULTIPLY BY x		MULTIPLY BY x^{-1}
2	$\frac{9}{16}x^{\frac{1}{3}}$	↔ product = integrand from 3 rd integration by parts (same as line above)
DIFFERENTIATE ↓	↓	ANTIDIFFERENTIATE
0	$\frac{27}{64}x^{\frac{4}{3}}$	↔ product = integrand from 4 th integration by parts

Now, we have a 0 in the u column,
 which means we are done with the table,
 and can collect the downward diagonal products.
 But we must be careful which factors we multiply.
 Remember that the red lines above correspond to rewriting the same integral, not to an integration by parts.
 So our downward diagonal products must never involve a factor from above a red line with a factor below the red line. (★)
 That means,

$$\int x^{\frac{1}{3}} (\ln x)^2 dx = (\ln x)^2 \left(\frac{3}{4} x^{\frac{4}{3}} \right) - 2(\ln x) \left(\frac{9}{16} x^{\frac{4}{3}} \right) + 2 \left(\frac{27}{64} x^{\frac{4}{3}} \right) + C$$

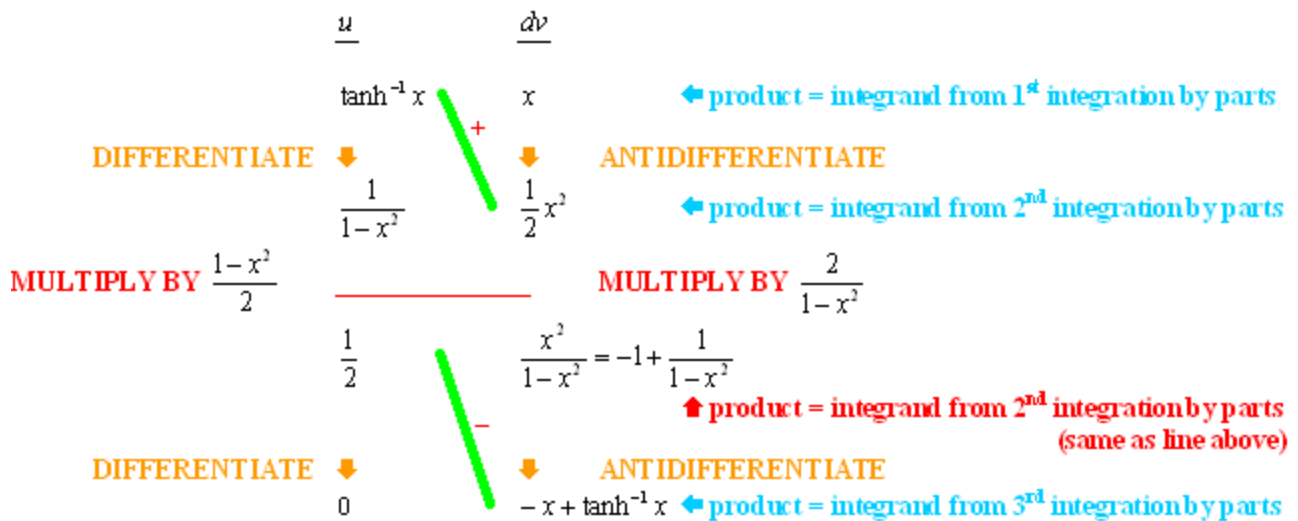
$$\int x^{\frac{1}{3}} (\ln x)^2 dx = \frac{3}{4} x^{\frac{4}{3}} (\ln x)^2 - \frac{9}{8} x^{\frac{4}{3}} \ln x + \frac{27}{32} x^{\frac{4}{3}} + C$$

$$\int x^{\frac{1}{3}} (\ln x)^2 dx = \frac{3}{32} x^{\frac{4}{3}} (8(\ln x)^2 - 12 \ln x + 9) + C$$

- (★) If you multiply a factor from above a red line with a factor below the red line,
 you are not multiplying u by v ,
 but rather u by $\frac{dv}{dx}$, which has nothing to do with integration by parts.

This technique is usually limited to specialized antiderivatives (like the original problem),
 and some complex situations in which integration by parts must be combined with another integration technique
 such as substitution, trigonometric/hyperbolic substitution, polynomial long division or partial fractions.

An example of such a situation is $\int x \tanh^{-1} x dx$:



$$\int x \tanh^{-1} x dx = \frac{1}{2} x^2 \tanh^{-1} x - \frac{1}{2} (-x + \tanh^{-1} x) + C$$

$$\int x \tanh^{-1} x dx = \frac{1}{2} x^2 \tanh^{-1} x + \frac{1}{2} x - \frac{1}{2} \tanh^{-1} x + C$$

$$\int x \tanh^{-1} x dx = \left(\frac{1}{2} x^2 - \frac{1}{2} \right) \tanh^{-1} x + \frac{1}{2} x + C$$

$$\int x \tanh^{-1} x dx = \frac{1}{2} ((x^2 - 1) \tanh^{-1} x + x) + C$$

An example of a much more complex situation is $\int x \cosh^{-1} x \, dx$:

	<u>u</u>		<u>dv</u>	
	$\cosh^{-1} x$	+	x	↖ product = integrand from 1 st integration by parts
DIFFERENTIATE	↓		↓	ANTIDIFFERENTIATE
	$\frac{1}{\sqrt{x^2-1}}$		$\frac{1}{2}x^2$	↖ product = integrand from 2 nd integration by parts
MULTIPLY BY	$\frac{x\sqrt{x^2-1}}{4}$			MULTIPLY BY
	$\frac{x}{4}$		$\frac{2x}{\sqrt{x^2-1}}$	↖ product = integrand from 2 nd integration by parts
DIFFERENTIATE	↓		↓	ANTIDIFFERENTIATE (SUBSTITUTION $u = x^2 - 1$)
	$\frac{1}{4}$	-	$\sqrt{x^2-1}$	(same as line above)
				↖ product = integrand from 3 rd integration by parts
DIFFERENTIATE	↓		↓	ANTIDIFFERENTIATE (SUBSTITUTION $x = \cosh t$)
	0	+	$\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\cosh^{-1} x$	↖ product = integrand from 4 th integration by parts

$$\int x \cosh^{-1} x \, dx = (\cosh^{-1} x) \left(\frac{1}{2} x^2 \right) - \frac{x}{4} \sqrt{x^2-1} + \frac{1}{4} \left(\frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} \cosh^{-1} x \right) + C$$

$$\int x \cosh^{-1} x \, dx = \frac{1}{2} x^2 \cosh^{-1} x - \frac{1}{4} x \sqrt{x^2-1} + \frac{1}{8} x \sqrt{x^2-1} - \frac{1}{8} \cosh^{-1} x + C$$

$$\int x \cosh^{-1} x \, dx = \left(\frac{1}{2} x^2 - \frac{1}{8} \right) \cosh^{-1} x - \frac{1}{8} x \sqrt{x^2-1} + C$$

$$\int x \cosh^{-1} x \, dx = \frac{1}{8} ((4x^2 - 1) \cosh^{-1} x - x \sqrt{x^2-1}) + C$$